Quintic periods and stability conditions via homological mirror symmetry

So Okada*

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Abstract

For the Fermat quintic Calabi-Yau threefold and the theory of stability conditions [Bri07], there have been two natural aims. One is that we should define central charges of stability conditions by quintic periods [CdGP] without losing quantum corrections. The other is that for well-motivated stability conditions on a derived Fukaya-type category, stable objects should be Lagrangians.

For the Fermat quintic Calabi-Yau threefold, we answer affirmatively to these aims with the simplest homological mirror symmetry in [Oka09, FutUed].

1 Introduction

Throughout this article, we say the quintic for the Fermat quintic Calabi-Yau threefold. Homological mirror symmetry (HMS for short) was introduced by Kontsevich [Kon95] to give a categorical framework of the mirror symmetry [CdGP], which most notably discusses the quintic. HMS states equivalences of triangulated categories of derived Fukaya-type categories and derived categories of coherent sheaves. HMS is a rapidly expanding subject. For example, in a recent paper [BDFKK], an unification of HMS and Mori program has been proposed for toric DM stacks.

Stability conditions [Bri07], which is inspired by Douglas' notion of Π-stabilities in superstring theory [Dou01, Dou02], is a categorical notion for triangulated categories. It is expected that for a derived Fukaya-type category and well-motivated stability conditions, each object of the derived category is uniquely decomposed into certain minimal Lagrangians. This is an origin of the notion [ThoYau, Tho]. This aim can be readily achieved for certain Fukaya-Seidel categories [Sei00, Sei01, Sei08], say, for ADE singularities, but we would like to relate stability conditions directly to quintic periods.

We would like to propose that HMS with the deformation theory of stability conditions indeed provides a categorical framework of the mirror symmetry.

*Partially supported by JSPS Grant-in-Aid #23740012; Email: okada@oyama-ct.ac.jp

Experts of stability conditions have considered the notion for projective varieties with *approximations* of periods [Dou02, Oka06] and have found profound applications such as [Bri08]. However, we certainly would like to construct stability conditions with central charges of quintic periods without approximations [Hos04, Kon12, KonSoi13] to better understand the mirror symmetry with HMS.

More explicitly, our view point is the following. HMS gives a set of Lagrangians such that the Ext-algebra of the Lagrangians makes the heart of a bounded t-structure of a triangulated category. We slightly relax the original notion of stability conditions. Our version is still categorical. With this notion, we prove that central charges of quintic periods near the Gepner point (the orbifold point) give stability conditions on the heart such that stable objects are Lagrangians vanishing cycles.

Notice that for our purposes, on a derived category of coherent sheaves, it does not matter whether we take an equivariance or not, as long as desired central charges give stability conditions in some manageable way. In fact, for the quintic, we use the simplest HMS, which takes an equivariant derived category of coherent sheaves of the quintic [Oka09, FutUed, IshUed]. Let us mention that for the quintic we have other HMS, which incorporate Novikov rings [NohUed, She].

The equivariant derived category has no non-trivial deformations, but stability conditions have ones, which locally make linear subspaces of linear functions from the Grothendieck group of the equivariant derived category to $\mathbb{C}$. In particular, we deal with deformations of stability conditions whose parameter is of the above Novikov rings.

Let us state the main theorem. Let $F = x_1^5 + \cdots + x_5^5 : \mathbb{C}^5 \to \mathbb{C}$, X be the quintic, $G \cong Z_5^5/Z_5$ be the group of multiplications of $\xi = \exp(\frac{2\pi i}{5})$ on the coordinates of $\mathbb{P}^4$, and $\mathrm{FS}(F)$ be the derived Fukaya-Seidel category of F. For central charges, we recall the following famous hypergeometric series:

$$\omega(x,p) = \sum_{n\geq 0} \frac{\Gamma(1+5(n+p))}{\Gamma(1+(n+p))^5} x^{n+p}.$$

For $F \in \mathrm{D}^b(\mathrm{Coh}\, X)$, the nilpotent element J of the second cohomology class of X, and $[1:x] \in \mathbb{P}^1$, we define central charges $Z_x(F)$ as the complex conjugate of the one in [Hos00, Hos04]:

$$\overline{Z_x(F)} = \int_X \mathrm{ch}(F) w(x, \frac{J}{2\pi i})\, \mathrm{Todd}\, X. \tag{1}$$

We put the complex conjugate so that the grading of Lagrangian vanishing cycles of $\mathrm{FS}(F)$ agree with the grading of semistable objects. However, the complex conjugate can be avoided by taking the dual of stability conditions as explained in Section 3.

In the sequel, we allow ourselves to freely recall explicit forms of period vectors, monodromy matrices, and a connection matrix in *loc cite*.

Let us mention that Mukai vectors give the first-order approximation of the right-hand side of Equation 1 near the large complex structure limit, leaving aside quantum corrections. For objects F of the equivariant derived category

$\mathrm{D}^b_G(\mathrm{Coh}\, X)$, we define $Z_x(F)$ with Equation 1 by forgetting the equivariance. We have the following.

Theorem 1.1. *For $\mathrm{D}^b_G(\mathrm{Coh}\, X) \cong \mathrm{FS}(F)$, the heart $\mathcal{A}$ of a bounded t-structure, and central charges Z_x near the Gepner point $x = \infty$, we have stability conditions on $\mathcal{A}$ such that stable objects are Lagrangian vanishing cycles.*

Let us recall that on stability conditions, we have the action of dilation and rotation, which is given by the complex multiplication on central charges [Oka06]. This in particular gives rise to tiltings of t-structures attached to stability conditions. For example, the gauge freedom [KleThe] gives a factor on central charges.

Near the Gepner point, in Lemma 4.2 we have stability conditions, which are, up to dilation and rotation, asymptotically the same as ones in Theorem 1.1. We call these stability conditions in Lemma 4.2 as stability conditions near the Gepner point as well. Near the large complex structure limit, we have stability conditions in Theorem 4.1 and we regard Lagrangian vanishing cycles in Theorem 1.1 as equivariant sheaves of the Beilinson basis with shifts [Bei].

In [CdGP], we essentially compute the mirror map to obtain the famous instanton sequence 5, 2875, 609250, ... of GW invariants. We have period vectors $\Pi^\infty_B(x)$ and $\Pi_B(x)$ and the connection matrix N, which we define later, such that $N\Pi^\infty_B(x) = \Pi_B(x)$. A ratio of components of the period vector $\Pi^\infty_B(x)$ is the mirror map. Taking complex conjugates, components of these period vectors give central charges of objects for stability conditions near the large complex structure limit and stability conditions near the Gepner point.

Stability conditions in Theorems 1.1, 4.1, and Lemma 4.2 deform into each other without a wall-crossing as in Corollary 4.3. However, we notice that the connection matrix N is a wall-crossing. Each component of $\Pi_B(x)$ is a central charge of a stable object for stability conditions near the Gepner point in Lemma 4.2. A single component of Π^∞_B is a central charge of a stable object for stability conditions near the large complex structure limit in Theorem 4.1. In particular, up to dilation and rotation on stability conditions by the single component, we have that the mirror map is not a central charge of a stable object. We have the following.

Corollary 1.2. Up to dilation and rotation on stability conditions, the mirror map gives the wall-crossing of the connection matrix.

Let us mention that constructing stability conditions for the quintic has been of a significant interest [BMT, BBMT]. In the following, we construct stability conditions for the quintic, taking a different approach for the mirror symmetry.

2 HMS

Let us recall our HMS for the quintic. Let A_4 be the Dynkin quiver of A type with four vertices and one-way arrows. In [Oka06, FutUed], we have proved

that

$$\mathrm{D}^b_G(\mathrm{Coh}\, X) \cong \mathrm{D}^b(\mathrm{mod}\, A_4^{\otimes 5}) \cong \mathrm{FS}(F).$$

Vertices of $A_4^{\otimes 5}$ are indexed by tuples of numbers $s = \{s_1, s_2, s_3, s_4, s_5\}$ for $0 \leq s_i \leq 3$ such that the source and sink vertices are indexed as $\{0, 0, 0, 0, 0\}$ and $\{3, 3, 3, 3, 3\}$. Each simple representation with one-dimensional complex vector space at the vertex s is also denoted by s. The quiver $A_4^{\otimes 5}$ has commuting relations on arrows. It is a Koszul duality that the Ext-algebra of simple representations s is anti-commutative.

In a regular zero locus of a morsification of $F : \mathbb{C}^5 \to \mathbb{C}$, Lagrangian vanishing cycles generate the derived Fukaya-Seidel category $\mathrm{FS}(F)$ with their Lagrangian Floer theory. The reader can consult [Aur] for an introduction to derived Fukaya-type categories.

For a morsification of F, simple objects s and its Ext-algebra correspond to Lagrangian vanishing cycles and their Lagrangian Floer theory. This is not very difficult to see, since $\mathrm{FS}(F)$ has no non-trivial deformations. Lagrangian vanishing cycles corresponding to s are graded with the lexicographic ordering of s. Also, let us mention that the Lagrangian Floer theory is categorically of zero-dimensional Lagrangians.

Let us recall that for $\mathrm{D}^b(\mathrm{Coh}\, X)$, we have the autoequivalence τ of the monodromy around the Gepner point such that $\tau^5 \cong [2]$. In particular, for the n-th exterior product of the cotangent bundle of $\mathbb{P}^4$ restricted to the quintic, denoted by Ω^n, we have $\mathcal{O}_X[3], \tau(\mathcal{O}_X[3]) \cong \mathcal{O}_X(1)[1], \tau^2(\mathcal{O}_X[3]) \cong \Omega(2)[2], \tau^3(\mathcal{O}_X[3]) \cong \Omega^2(3)[3], \tau^4(\mathcal{O}_X[3]) \cong \Omega^3(4)[4]$ [Asp], which makes the following quintic quiver [DGJT] with Ext^1 arrows and the dashed arrow indicating $[-2]$:

$$\begin{tikzcd}
 & \mathcal{O}_X[3] \arrow[ddl, dashed, "{[-2]}"', "\mathrm{Ext}^1"] & \\
 & & \tau(\mathcal{O}_X[3]) \cong \mathcal{O}_X(1)[1] \arrow[ul, "\mathrm{Ext}^1"'] \\
\tau^4(\mathcal{O}_X[3]) \cong \Omega^3(4)[4] \arrow[ddr, "\mathrm{Ext}^1"] & & \\
 & & \tau^2(\mathcal{O}_X[3]) \cong \Omega(2)[2] \arrow[uu, "\mathrm{Ext}^1"'] \\
 & \tau^3(\mathcal{O}_X[3]) \cong \Omega^2(3)[3] \arrow[ur, "\mathrm{Ext}^1"] &
\end{tikzcd}$$

Figure 1: Quintic quiver

Objects $\tau^i(\mathcal{O}_X)$ are of the Beilinson basis with shifts. By Orlov's category of graded matrix factorizations of F [Orl], we can check that by forgetting equivariance, simple representations s of $\mathrm{mod}\, A_4^{\otimes 5}$ give $\tau^{-\sum s_i}(\mathcal{O}_X)$.

3 Stability conditions

Let us modify the notion of stability conditions [Bri07] for our purposes. By a stability condition on a triangulated category, each nonzero object is decomposed into semistable objects, uniquely up to isomorphisms, by the finite sequence of exact triangles.

We do not recall the full definition of the stability conditions. Instead, we recall that to give a stability condition on a triangulated category is equivalent to give the heart of a bounded t-structure of the triangulated category and a central charge, which is a linear function from the Grothendieck group of the triangulated category to $\mathbb{C}$, with the so-called Harder-Narasimhan property defined below.

The original definition of stability condition [Bri07] requires that central charges of nonzero objects of the heart are contained in the upper-half plane plus the negative real line. We omit this condition. For semistable objects, we simply ask the compatibility between phases and angles of central charges. Let us define the following.

Definition 3.1. For a triangulated category $\mathcal{T}$ and the Grothendieck group $K(\mathcal{T})$, a stability condition consists of the heart $\mathcal{A}$ of a bounded t-structure of $\mathcal{T}$ and a central charge $Z \in \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{T}), \mathbb{C})$ with the following properties.

1. We have *semistable objects* $A \in \mathcal{A}$ such that for $Z(A) = m(A)\exp(\phi_A i)$ of *masses* $m(A) > 0$ and *phases* $\phi_A \in \mathbb{R}$.
2. If $\phi_{A'} > \phi_A$, we have $\mathrm{Hom}(A', A) \cong 0$.
3. For each nonzero object $E \in \mathcal{A}$, we have semistable objects $A_i \in \mathcal{A}$ such that $\phi_{A_{i+1}} > \phi_{A_i}$ and we have the following filtration by short exact sequences:

$$\begin{tikzcd}
0 = E_n \arrow[r] & E_{n-1} \arrow[r] \arrow[dl] & E_{n-2} \arrow[r] \arrow[dl] & \dots \arrow[r] & E_1 \arrow[r] \arrow[d] & E_0 = E \arrow[d] \\
A_{n-1} & A_{n-2} & & & A_1 & A_0
\end{tikzcd}$$

For each phase, semistable objects which can not be obtained by non-trivial extensions of semistable objects of the phase are called *stable objects*. The above filtration is called *the Harder-Narasimhan filtration* of E and having such filtrations for non-zero objects of the heart, originally for central charges of semistable objects in the upper-half plane plus the negative real line, is called the Harder-Narasimhan property of the central charge on the heart.

By the second and the fourth conditions in Definition 3.1, we have the uniqueness of the Harder-Narasimhan filtrations and families of bounded t-structures attached to stability conditions [GKR]. Deformation of stability conditions is given by suitable deformation of central charges as in [Bri07].

For semistable objects $A, A' \in \mathcal{A}$, in some cases, $|\phi_A - \phi_{A'}|$ can indeed be arbitrary large and $\mathrm{Ext}^1(A, A')$ can give a semistable object after a wall-crossing by changing the sign of $\phi_A - \phi_{A'}$.

We can also take the dual of Definition 3.1 by taking the opposite inequalities in the second and the third conditions in Definition 3.1. If we take this dual of Definition 3.1, we can get rid of the complex conjugate in Equation 1.

4 Statements and proofs

Let us prove our statements. For each simple representation s of $\mathrm{mod}\, A_4^{\otimes 5}$ and $[1 : x]$, we have its central charge as

$$Z_x(s) = Z_x(\tau^{-\sum s_i}(\mathcal{O}_X)).$$

In the sequel, as the heart $\mathcal{A}$ of a bounded t-structure of $\mathrm{D}^b_G(\mathrm{Coh}\, X) \cong \mathrm{FS}(F)$, we take $\mathrm{mod}\, A_4^{\otimes 5}$. Let us give a proof of Theorem 1.1.

Proof. For $k = 1, \cdots, 5$ and $j = 0, 1, \cdots, 4$, we have

$$\tilde{\omega_k}(x) = -\frac{1}{5}\frac{1}{(2\pi i)^4}\sum_{N=0}^{\infty}\frac{\Gamma(N+\frac{k}{5})^5}{\Gamma(5N+k)}x^{-N-\frac{k}{5}},$$

$$\omega_j^{\infty}(x) = \sum_{k=1}^{5}(1-\xi^k)^4\xi^{kj}\tilde{\omega_k}(x).$$

Let $\Pi_B^{\infty}(x) = {}^t(\omega_0^{\infty}(x), \omega_1^{\infty}(x), \omega_2^{\infty}(x), \omega_4^{\infty}(x))$ be a period vector around the Gepner point.

For the following connection matrix

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{2}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{21}{5} & \frac{1}{5} & \frac{3}{5} & -\frac{8}{5} \\ 1 & -1 & 0 & 0 \end{pmatrix},$$

the fourth component of the vector $N\Pi_B^{\infty}(x)[4]$ is $Z_x(\mathcal{O}_X)$. The monodromy matrix of the period vector around the Gepner point is the following.

$$M_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

For large x and $a = \mathrm{Arg}(N\Pi_B^{\infty}(x)[4])$, we can take $\mathrm{Arg}(NM\Pi_B^{\infty}(x)[4])$, $\mathrm{Arg}(NM^2\Pi_B^{\infty}(x)[4])$, $\mathrm{Arg}(NM^3\Pi_B^{\infty}(x)[4])$, $\approx a + \frac{2\pi}{5}$, $a + 2\frac{2\pi}{5}$, $a + 3\frac{2\pi}{5}$, $a + 4\frac{2\pi}{5}$.

Let our stable objects be objects s and their phases be angles of $\overline{Z_x(s)}$ in the increasing way approximately by $\frac{2\pi}{5}$ as $\sum a_i$ increases.

It is clear from the standard theory of quiver representations that these stable objects together with their self-direct sums taken as semistable objects give stability conditions. $\square$

A stability condition at the Gepner point would not exist, unless we take the action of dilation and rotation. Masses of central charges of S tend to be zero and the same as $x \to \infty$. The notion of stability conditions whose central charges have the pentagon symmetry has been discussed in [Oka09, Tod]. Let us take an object in $\mathrm{D}^b(\mathrm{Coh}\, X)$ such that its equivariant object is stable. We obtain a point moduli. By counting its extensions by DT invariants [KonSoi08], we obtain a quasimodular form (a mock modular form) [KanZag] as in [Oka09, MelOka].

Let us turn to the large complex structure limit. In this case, as explained, we regard objects s as equivariant objects of the Beilinson basis with shifts. For coherent sheaves of X, near the large complex structure limit, let us mention that it is known that how their central charges approximately map into $\mathbb{C}$ depending on dimensions of their supports. Let us state the following.

Theorem 4.1. *For the equivariant derived category* $\mathrm{FS}(F) \cong \mathrm{D}^b_G(\mathrm{Coh}\, X)$*, the heart* $\mathcal{A}$ *of a bounded t-structure of* $\mathrm{D}^b_G(\mathrm{Coh}\, X)$*, and central charges* Z_x *near the large complex structure limit* $x = \infty$*, we have stability conditions on* $\mathcal{A}$ *such that stable objects are equivariant objects of the Beilinson basis with shifts.*

Proof. Let us recall the following.

$$\begin{aligned}
w^{(0)}(x) &= w(x,0),\\
w^{(1)}(x) &= \frac{1}{2\pi i}\frac{\partial}{\partial p} w(x,p)\mid_{p=0},\\
w^{(2)}(x) &= \frac{1}{2!(2\pi i)^2}\; 5\; \frac{\partial^2}{\partial p^2} w(x,p)\mid_{p=0} + \frac{11}{2}\frac{1}{2\pi i}\frac{\partial}{\partial p} w(x,p)\mid_{p=0},\\
w^{(3)}(x) &= -\frac{1}{3!(2\pi i)^3}\; 5\; \frac{\partial^3}{\partial p^3} w(x,p)\mid_{p=0} - \frac{1}{2\pi i}\;\frac{50}{12}\;\frac{\partial}{\partial p} w(x,p)\mid_{p=0}.
\end{aligned}$$

Then we have a period vector around the large complex structure limit $\Pi_B(x) = {}^t(w^{(0)}(x), w^{(1)}(x), w^{(2)}(x), w^{(3)}(x))$, which is $N\Pi_B^\infty(x)$. For the following monodromy matrix M_∞ of $\Pi_B(x)$ around the Gepner point

$$M_\infty = \begin{pmatrix} 1 & 0 & 0 & -1\\ -1 & 1 & 0 & 1\\ -3 & -5 & 1 & 3\\ 5 & -8 & 1 & -4 \end{pmatrix},$$

the fourth component of the vector $\Pi_B^\infty(x)[4]$ is $Z_x(\mathcal{O}_X)$.

For small x, we can take

$$a_i = \mathrm{Arg}(M_\infty^{i+1}\Pi_B(x)[4]) - \mathrm{Arg}(M_\infty^{i}\Pi_B(x)[4])$$

to be $\pi > a_0 > a_1 > a_2 > a_3 \underline{> 0}$ and $a_i \approx \pi$. Let us take objects s as stable objects and let angles of $\overline{Z_x(s)}$ be their phases in the increasing way approximately by π as $\sum a_i$ increases. ∎

Similarly, a stability condition at the large complex structure limit would not exist, unless we take the action of dilation and rotation. For small x,

$F_i = |M_\infty^i \Pi(x)[4]|$ tends to be infinitely large and satisfy $4F_0 \approx F_1$,$6F_0 \approx F_2$, $F_1 \approx F_3$, and $F_0 \approx F_4$. A stability condition at the large complex structure limit would be invariant under the action of adding the central charge $\Pi(x)[1]$, which is of a skyscraper sheaf of X, to $M_\infty^i \Pi(x)[4]$, since $|\Pi(x)[1]|/F_0$ tends to be zero as $x \to 0$.

To state the following corollaries, let us recall the following. There is the Picard-Fuchs equation:

$$\{(x\frac{d}{dx})^4 - 5^5 x(x\frac{d}{dx} + \frac{4}{5})(x\frac{d}{dx} + \frac{3}{5})(x\frac{d}{dx} + \frac{2}{5})(x\frac{d}{dx} + \frac{1}{5})\}(\ \cdot\ ) = 0.$$

Let us put the following anti-symplectic integer matrix N', which is the multiplication of the inverse of the symplectic integer matrix $N_{k=5}$ in the Equation (5.4) in [KleThe] and the diagonal matrix below:

$$N' = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is been noted in [Hos00] that $N'\Pi_B(x)$ is a period vector $\vec{\Pi}'$ in [KleThe].

Near the Gepner point, let us define central charges Z'_x on the heart $\mathcal{A}$ as follows.

$$\overline{Z'_x(s)} = M^{-\sum s_i} \Pi_B^\infty(x)[4].$$

Lemma 4.2. On the heart $\mathcal{A}$, central charges Z'_x define stability conditions which are, up to a rotation, asymptotically the same as the ones defined in the proof of Theorem 1.1 with respect to the metric on stability conditions defined in [Bri07].

Proof. We take s to be stable objects and define their phases in the same way as in the proof of Theorem 1.1. For each k, $\mathrm{Arg}\left(\frac{NM^k \Pi_B^\infty(x)[4]}{NM^{k+1}\Pi_B^\infty(x)[4]}\right) -$ $\mathrm{Arg}\left(\frac{M^k \Pi_B^\infty(x)[4]}{M^{k+1}\Pi_B^\infty(x)[4]}\right)$ and $|NM^k\Pi_B^\infty(x)[4]| - |M^k\Pi_B^\infty(x)[4]|$ tend to be zero as $x \to 0$. □

For stability conditions in Theorems 1.1 and 4.1 and in Lemma 4.2, let us prove the following.

Corollary 4.3. Stability conditions in Theorems 1.1 and 4.1 and in Lemma 4.2 deform into each other without a wall-crossing.

Proof. From phases of central charges of stable objects in the proofs of Theorems 1.1 and 4.1 and of Lemma 4.2, we can deform central charges so that we have desired angles and masses of the same stable objects. □

Let us prove the following to obtain quintic periods from stability conditions.

Corollary 4.4. Near the Gepner point, we have one-parameter subspaces of the deformation space of stability conditions such that complex conjugates of maximally linearly independent central charges of stable objects solve the Picard-Fuchs equation and give quintic periods up to the multiplication by $N'N$.

Proof. We take stability conditions in Lemma 4.2. Other than $M_0^3\Pi_B^\infty(x)[4]$, which is $-\omega_0^\infty(x) - \omega_1^\infty(x) - \omega_2^\infty(x) - \omega_4^\infty(x)$, we have $M_0\Pi_B^\infty(x)[4] = \omega_0^\infty(x)$, $M_0^2\Pi_B^\infty(x)[4] = \omega_1^\infty(x)$, $M_0^3\Pi_B^\infty(x)[4] = \omega_2^\infty(x)$, and $\Pi_B^\infty(x)[4] = \omega_4^\infty(x)$. So, we can multiply central charges of stable objects $\Pi_B^\infty(x)$ by N and obtain $\Pi_B(x)$. $\square$

Let us give a proof of Corollary 1.2. The mirror map is $\frac{\omega^{(2)}(x)}{\omega^{(3)}(x)}$, which in the notation of [KleThe] is $\frac{\omega^1}{\omega^2}$ for $\omega^1 = \frac{\omega_2^2}{\mathcal{G}_2}\omega^{(2)}(x)$ and $\omega^2 = \frac{\omega_2^2}{\mathcal{G}_2}\omega^{(3)}(x)$. Let us compare period vectors $\Pi_B^\infty(x)$ and $\Pi_B(x)$ with stability conditions in Theorem 4.1 and Lemma 4.2.

Proof. For stability conditions near the large complex structure limit in Theorem 4.1, $\omega^{(2)}(x)$ is not a central charge of a stable object and $\omega^{(3)}(x)$ is a central charge of a stable object. For stability conditions in Lemma 4.2, each component is a central charge of a stable object. Up to dilation and rotation by $\omega^{(3)}(x)$ on stability conditions in Theorem 4.1, we have that $\frac{\omega^{(2)}(x)}{\omega^{(3)}(x)}$ is not a central charge of a stable object. $\square$

Stability conditions in Theorems 1.1 and 4.1 and in Lemma 4.2 are essentially of the notion of King's theta stabilities [Kin], on which the notion of stability conditions [Bri07] modeled. In fact, we can take some n-th roots of central charges of stable objects to obtain stability conditions in the form of [Bri07]. By doing so, however, our statements do not hold as stated.

Corollaries 4.3, 4.4, and 1.2 are nontrivial. To be able to deform stability conditions in a certain way, in general we have to carefully look into distributions of central charges of stable objects.

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